

LIE ALGEBRAS AND ADO'S THEOREM

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ABSTRACT. In this article, we begin by providing a detailed description of the basic definitions and properties of Lie algebras and their representations. Afterward, we prove a few important theorems, such as Engel's Theorem and Levi's Theorem, and introduce a number of tools, like the universal enveloping algebra, that will be required to prove Ado's Theorem. We then deduce Ado's Theorem from these preliminaries.

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The author hereby affirms his awareness of the standards of the Harvard College Honor Code.

1. MOTIVATIONS AND DEFINITIONS

1.1. **Historical Background.**¹

The vast and beautiful theory of Lie groups and Lie algebras has its roots in the work of German mathematician Christian Felix Klein (1849–1925), who sought to describe the geometry of a space, such as a real or complex manifold, by studying its group of symmetries. But it was his colleague, the Norwegian mathematician Marius Sophus Lie (1842–1899), who had the insight to study the action of symmetry groups on manifolds infinitesimally as a means of determining the action locally. Lie thus created and developed the theory of continuous symmetry that we now call “Lie theory” in his honor, and his work has had profound consequences in a variety of fields, including particle physics, where the notion of Lie algebra representation is fundamental to the study of elementary particles.

A natural question that arises in the study of group representations is whether or not a given group is linear, in that it admits a faithful finite-dimensional representation (here the answer is yes for finite groups, because Cayley’s theorem tells us that every finite group is isomorphic to a subgroup of a symmetric group, but counterexamples may be readily found among infinite groups). One can ask the analogous question in the context of Lie algebras: does every finite-dimensional (real or complex) Lie algebra admit a faithful finite-dimensional representation? Lie had long suspected the answer to be in the affirmative, but he was unable to provide a complete proof; the statement was first proven by Russian mathematician Igor Dmitrievich Ado (1910–1983), a student of Chebotarev, as part of his doctoral dissertation. As it happens, the faculty at Kazan State University, where Ado was a student, were so impressed with his work that instead of granting him a doctorate, they awarded him the degree of habilitation, which is the highest academic degree offered by many universities throughout Eurasia. Since the work of Ado, his theorem has been generalized by Iwasawa and Harish-Chandra to hold for Lie algebras over fields with arbitrary characteristic.

In this paper, we will start by providing a detailed description of the basic definitions and properties of Lie algebras and their representations. Afterward, we will prove a number of important theorems that will serve as key stepping-stones in the last section of the paper, where we will detail a complete proof of Ado’s Theorem.

1.2. Defining Lie Algebras. We begin by defining Lie algebras abstractly, in the sense that our definition makes no reference to Lie groups. The motivation for studying Lie algebras independently of Lie groups has its origins in the work

¹See [5] for a more comprehensive treatment on the history of Lie Theory, and see [6] for a more detailed discussion on the history of Ado’s Theorem.

of German mathematician Wilhem Karl Joseph Killing (1847–1923), who had no access to the Scandinavian journals that Lie published in and consequently invented the theory of Lie algebras on his own (for which he received much scorn from Lie).

Definition 1. An *abstract Lie algebra* \mathfrak{g} is a (real or complex) vector space equipped with an antisymmetric bilinear form $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, known as a *Lie bracket*, that satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

The Jacobi identity may appear to be strange (and unmotivated) at first glance, but it has a number of important formal consequences and demonstrates that abstract Lie algebras are not generally associative: indeed, we have that $[[X, Y], Z] - [X, [Y, Z]] = [Y, [Z, X]]$.

We say that a Lie algebra is *finite-dimensional* if its underlying vector space is finite-dimensional. Given that an abstract Lie algebra has the underlying structure of vector space, it is natural to wonder whether there are notions analogous to vector subspaces, direct sums, and homomorphisms. The key to answering these questions is to determine how these structures should interact with the Lie bracket. For the case of vector subspaces, the answer depends on how strongly we want the Lie bracket structure to be preserved:

Definition 2. Let \mathfrak{g} be an abstract Lie algebra. Then a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is said to be a *Lie subalgebra* of \mathfrak{g} if \mathfrak{h} is itself closed under the Lie bracket operation (i.e. $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$, or simply $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$). A Lie subalgebra \mathfrak{h} of \mathfrak{g} is said to be an *ideal* of \mathfrak{g} if \mathfrak{h} is closed under taking Lie brackets with arbitrary elements of \mathfrak{g} (i.e. $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, or simply $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$).

As for direct sums, the definition is exactly what one would expect, with the Lie bracket operating component-wise:

Definition 3. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be abstract Lie algebras. Then the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ naturally has the structure of Lie algebra, with the Lie bracket given by $[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2])$ for all $X_1, Y_1 \in \mathfrak{g}_1$ and $X_2, Y_2 \in \mathfrak{g}_2$.

Note that if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a direct sum decomposition of a Lie algebra \mathfrak{g} into two Lie subalgebras $\mathfrak{g}_1, \mathfrak{g}_2$, then $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$, in the sense that $[X, Y] = 0$ for all $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_2$; it follows that $\mathfrak{g}_1, \mathfrak{g}_2$ are in fact ideals of \mathfrak{g} . Finally, Lie algebras should “talk to each other” in a way that is compatible with the Lie bracket operation:

Definition 4. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be abstract Lie algebras. A map of vector spaces $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism if $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}_1$.

Before moving on to defining Lie algebra representations, we pause to consider a number of important examples that should help contextualize the heretofore-described concepts:

Example 5. The following seven points not only illustrate the concepts introduced in Definitions 1–4, but also present important observations and tools that will be used in the rest of this article:

- (a) Given any (real or complex) vector space V , we can turn V into a Lie algebra by equipping it with the trivial bracket $[X, Y] = 0$ for all $X, Y \in V$; such a Lie algebra is said to be *abelian*. In fact, any 1-dimensional subspace of a Lie algebra is an abelian subalgebra: if X is a nonzero element of such a subspace, then $[aX, bX] = ab[X, X] = 0$ for all scalars a, b . Note that any subspace of an abelian Lie algebra is an ideal.
- (b) The *center* $Z_{\mathfrak{g}}$ of a Lie algebra \mathfrak{g} is the ideal of \mathfrak{g} defined by $Z_{\mathfrak{g}} = \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$. That $Z_{\mathfrak{g}}$ is an ideal is an immediate consequence of the Jacobi identity: indeed, we have that $[[X, Y], Z] = [X, [Y, Z]] + [Y, [Z, X]] = 0 + 0 = 0$ for all $X, Y \in Z_{\mathfrak{g}}$ and $Z \in \mathfrak{g}$. If \mathfrak{g} is abelian then $Z_{\mathfrak{g}} = \mathfrak{g}$.
- (c) The Lie algebra $\mathfrak{gl}_n(k)$, where $k = \mathbb{R}$ or $k = \mathbb{C}$, is called the *general linear Lie algebra*, and it is defined to be the space of $n \times n$ matrices with entries in k , with the Lie bracket given by the commutation of matrices $[X, Y] = XY - YX$ for all $X, Y \in \mathfrak{gl}_n(k)$ (note that $\mathfrak{gl}_n(k)$ is nonabelian). It contains, as a Lie subalgebra, the *special linear Lie algebra* $\mathfrak{sl}_n(k)$ whose elements are traceless matrices — note here that the trace of the commutation of any two matrices in $\mathfrak{gl}_n(k)$ is 0, so $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] \subset \mathfrak{sl}_n(k)$; in particular, $[\mathfrak{sl}_n(k), \mathfrak{gl}_n(k)] \subset \mathfrak{sl}_n(k)$, so $\mathfrak{sl}_n(k)$ is an ideal of $\mathfrak{gl}_n(k)$.
- (d) More abstractly, given a k -vector space V , we obtain the Lie algebra $\mathfrak{gl}(V)$ of k -endomorphisms of V , with the Lie bracket given by $[S, T] = S \circ T - T \circ S$ for all $S, T \in \mathfrak{gl}(V)$. If V is finite-dimensional, identifying V with k^n by choosing a basis induces a corresponding identification of $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n(k)$.
- (e) Note that if \mathfrak{g} is an abstract Lie algebra and \mathfrak{h} is a Lie subalgebra, the inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is a Lie algebra homomorphism. Also, if $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism, then the kernel $\ker \phi \subset \mathfrak{g}_1$ is an ideal because we have that $\phi([X, Y]) = [\phi(X), \phi(Y)] = [0, \phi(Y)] = 0$ for all $X \in \ker \phi$ and $Y \in \mathfrak{g}_1$.
- (f) Let \mathfrak{g} be an abstract Lie algebra, and let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. Then one readily verifies that the quotient vector space $\mathfrak{g}/\mathfrak{a}$ has the natural structure of Lie algebra. Given a Lie algebra homomorphism $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$, the induced map of vector spaces $\mathfrak{g}_1/\ker \phi \rightarrow \mathfrak{g}_2$ is also a Lie algebra homomorphism.

- (g) Let \mathfrak{g} be a real Lie algebra. Then the vector space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus (i \cdot \mathfrak{g})$ is a complex Lie algebra under the complex-linear Lie bracket operation defined by $[(X, iY), (X', iY')] = [(X, X') - (Y, Y'), (X, Y') + (Y, X')]$. In this setting, we say that $\mathfrak{g}_{\mathbb{C}}$ is the *complexification* of \mathfrak{g} and that \mathfrak{g} is a *real form* for $\mathfrak{g}_{\mathbb{C}}$. ♣

We shall expand our discussion of the properties of Lie algebras in Section 2.1.

1.3. Defining Representations of Lie Algebras. Representation theory is the study of how algebraic objects, like groups and Lie algebras, act on vector spaces. We now turn our attention to defining representations of Lie algebras; the first major strides in this subject were made by French mathematician Élie Joseph Cartan (1869–1951), who classified the irreducible finite-dimensional representations of simple Lie algebras.

Definition 6. Let V be a (real or complex) vector space, and let \mathfrak{g} be an abstract Lie algebra. A (real or complex) *Lie algebra representation* is a Lie algebra homomorphism $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. Equivalently, a (real or complex) Lie algebra representation is a bilinear map $\mathfrak{g} \times V \rightarrow V$, denoted by $(X, v) \mapsto X \cdot v$, satisfying $[X, Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v)$ for all $X, Y \in \mathfrak{g}$ and $v \in V$.²

We say that a Lie algebra representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is *finite-dimensional* if V is finite-dimensional and is *faithful* if μ is injective. When referring to representations, we shall interchangeably write μ (the map) or V (the vector space on which \mathfrak{g} acts). Any reader familiar with the representation theory of groups will wonder whether the notions of invariant subspace, direct sum, and representation homomorphism carry over to the representation theory of Lie algebras, and indeed they do. For invariant subspaces, we have:

Definition 7. Let $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of an abstract Lie algebra \mathfrak{g} . A vector subspace $W \subset V$ is said to be *\mathfrak{g} -invariant* if $\mathfrak{g} \cdot W \subset W$, in the sense that $X \cdot w \in W$ for all $X \in \mathfrak{g}$ and $w \in W$. In this case, W gives rise to a *subrepresentation* $\mathfrak{g} \rightarrow \mathfrak{gl}(W)$ of V , defined by restricting the action of \mathfrak{g} on V to W .

Direct sums of Lie algebra representations are defined in the obvious way, with the Lie algebra elements acting block-diagonally:

Definition 8. Let $\mu_1 : \mathfrak{g} \rightarrow \mathfrak{gl}(V_1)$ and $\mu_2 : \mathfrak{g} \rightarrow \mathfrak{gl}(V_2)$ be representations of an abstract Lie algebra \mathfrak{g} . Then the map $\mu_1 \oplus \mu_2 : \mathfrak{g} \rightarrow \mathfrak{gl}(V_1 \oplus V_2)$ defined by $X \cdot (v_1, v_2) = (X \cdot v_1, X \cdot v_2)$ for all $X \in \mathfrak{g}$, $v_1 \in V_1$, and $v_2 \in V_2$ is a Lie algebra representation.

²The two definitions are related by $X \cdot v = \mu(X)(v)$ for all $X \in \mathfrak{g}$ and $v \in V$. We shall use the notation interchangeably.

A Lie algebra representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is said to be *irreducible* if its only \mathfrak{g} -invariant subspaces are 0 and V . This terminology suggests that we may be able to split a non-irreducible representation as a direct sum of reducible components. Lie algebra representations satisfying this property are given a special name:

Definition 9. Let $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a Lie algebra representation. Then μ is said to be *semisimple* if for every \mathfrak{g} -invariant subspace $W \subset V$ there exists a \mathfrak{g} -invariant subspace $W' \subset V$ such that $V = W \oplus W'$.

Note that in the setting of Definition 9, we have $V = W \oplus W'$ as Lie algebra representations, in the sense that the equality is compatible with the \mathfrak{g} -action. Finally, Lie algebra representations should “talk to each other” in a way that is compatible with the action of the Lie algebra elements:

Definition 10. Let $\mu_1 : \mathfrak{g} \rightarrow \mathfrak{gl}(V_1)$ and $\mu_2 : \mathfrak{g} \rightarrow \mathfrak{gl}(V_2)$ be representations of an abstract Lie algebra \mathfrak{g} . A map $\phi : V_1 \rightarrow V_2$ of vector spaces is said to be a *\mathfrak{g} -representation homomorphism* if $\phi(X \cdot v) = X \cdot \phi(v)$ for all $X \in \mathfrak{g}$ and $v \in V_1$.

We conclude this section with an important example delineating the concepts introduced above:

Example 11. The following four points illustrate the concepts introduced in Definitions 6–10:

- (a) A Lie algebra representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is said to be *trivial* if μ is the 0-homomorphism. The endomorphism algebra $\mathfrak{gl}(V)$ of a vector space is naturally a representation of itself via the identity map.
- (b) Every Lie algebra \mathfrak{g} is automatically equipped with its *adjoint representation* $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$, defined by $\text{ad}(X) \cdot Y = [X, Y]$. That ad is a well-defined representation follows immediately from the Jacobi identity. The adjoint representation is fundamental to the study of Lie algebras. Note that $Z_{\mathfrak{g}} = \ker \text{ad}(\mathfrak{g})$ and in particular that the adjoint representation of an abelian Lie algebra is trivial.
- (c) The map $\mu : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}_n(k)$ given by choosing a basis to identify a finite-dimensional vector space V with k^n (see part (d) of Example 5) is known as the *standard representation* of $\mathfrak{gl}(V)$ on k^n . Note that this representation is irreducible, because every nonzero vector in k^n is carried by some endomorphism to any other vector in k^n .
- (d) Not every Lie algebra representation is semisimple. Let $\mathfrak{g} = \mathbb{R}$ and $V = \mathbb{C}^2$, and consider the representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ defined by $\mu(1) = e_{1,2}$ (the elementary matrix with row-1, column-2 entry equal to 1 and all other entries equal to 0). The nonzero proper \mathfrak{g} -invariant subspaces of V must be spanned by eigenvectors of $e_{1,2}$, but the vector $(1, 0) \in \mathbb{C}^2$ is the only eigenvector of $e_{1,2}$. Thus, V has only one nonzero

proper \mathfrak{g} -invariant subspace and hence cannot be decomposed into a direct sum of irreducible subrepresentations. ♣

2. MORE ON LIE ALGEBRAS AND THEIR REPRESENTATIONS

2.1. Properties of Lie Algebras. In part (a) of Example 5, we snuck in the definition of an abelian Lie algebra, one in which the Lie bracket is trivial. In this subsection, we will discuss the failure of abstract Lie algebras to be abelian as a means of characterizing them. It will be prudent to start with a quick but useful lemma on ideals:

Lemma 12. *Let \mathfrak{g} be an abstract Lie algebra, and let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ be ideals. Then $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $[\mathfrak{a}, \mathfrak{b}]$ are also ideals of \mathfrak{g} . Moreover, we have that $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \simeq \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$.*

Proof. That $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are ideals is obvious; that $[\mathfrak{a}, \mathfrak{b}]$ satisfies $[[\mathfrak{a}, \mathfrak{b}], \mathfrak{g}] \subset [\mathfrak{a}, \mathfrak{b}]$ follows immediately from the Jacobi identity (see part (b) of Example 5 for inspiration). The only tricky part is to show that $[\mathfrak{a}, \mathfrak{b}]$ is a vector subspace of \mathfrak{g} , but observe that for any $W, Y \in \mathfrak{a}$ and $X, Z \in \mathfrak{b}$ we have $[W, X] + [Y, Z] = [W + Y, X + Z] - [W, Z] - [Y, X] \in [\mathfrak{a}, \mathfrak{b}]$, and closure under scalar multiplication is manifest. The composite map $\mathfrak{a} \hookrightarrow \mathfrak{a} + \mathfrak{b} \twoheadrightarrow (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is surjective and has kernel $\mathfrak{a} \cap \mathfrak{b}$, so we have an isomorphism of vector spaces $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \simeq \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$; this isomorphism is a Lie algebra homomorphism by part (f) of Example 5. ♠

We now define two more general types of Lie algebras, which will be characterized by the extent to which they fail to be abelian:

Definition 13. A Lie algebra \mathfrak{g} is said to be *nilpotent* if some term (and hence all subsequent terms) of the sequence $(\mathfrak{g}_n : n \in \mathbb{N})$ defined by $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_n = [\mathfrak{g}_{n-1}, \mathfrak{g}]$ for all $n \geq 1$:

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \supset [[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}], \mathfrak{g}] \supset \dots,$$

which is called the *lower central series*, is equal to 0. More generally, \mathfrak{g} is said to be *solvable* if some term (and hence all subsequent terms) of the sequence $(\mathfrak{g}^n : n \in \mathbb{N})$ defined by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}^{n-1}]$ for all $n \geq 1$:

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supset [[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]] \supset \dots,$$

which is called the *derived series*, is equal to 0.

By Lemma 12, every term in the lower central and derived series of a Lie algebra is an ideal. Solvable Lie algebras are “more general” than nilpotent Lie algebras in the sense that every nilpotent Lie algebra is solvable: indeed, one may check (via induction) that $\mathfrak{g}^n \subset \mathfrak{g}_n$ for all $n \geq 0$, so $\mathfrak{g}_n = 0$ for some n implies that $\mathfrak{g}^n = 0$ as well. The following example illustrates the difference between solvable and nilpotent Lie algebras:

Example 14. Let $\mathfrak{b}_n(k) \subset \mathfrak{gl}_n(k)$, where $k = \mathbb{R}$ or $k = \mathbb{C}$, denote the Lie subalgebra of upper triangular $n \times n$ matrices with entries in k , and let $\mathfrak{n}_n(k) \subset \mathfrak{b}_n(k)$ denote the Lie subalgebra whose elements are strictly upper triangular matrices. We will show that \mathfrak{b}_n is solvable but not nilpotent, whereas \mathfrak{n}_n is in fact nilpotent and therefore also solvable.

By definition, we have that $(\mathfrak{b}_n(k))^0 = (\mathfrak{b}_n(k))_0 = \mathfrak{b}_n(k)$. Now $(\mathfrak{b}_n(k))^1 = (\mathfrak{b}_n(k))_1 = [\mathfrak{b}_n(k), \mathfrak{b}_n(k)]$, and the commutator of any two upper-triangular matrices is strictly upper-triangular. Thus, we have that $(\mathfrak{b}_n(k))^1 = (\mathfrak{b}_n(k))_1 \subset \mathfrak{n}_n(k)$. If $e_{i,j}$ denotes the elementary matrix with 1 in the row- i , column- j entry and 0 everywhere else, then observe that for $i < j$ we have $[e_{i,j}, e_{j,j}] = e_{i,j}$, from which it follows that $(\mathfrak{b}_n(k))^1 = (\mathfrak{b}_n(k))_1 \supset \mathfrak{n}_n(k)$, implying that $(\mathfrak{b}_n(k))^1 = (\mathfrak{b}_n(k))_1 = \mathfrak{n}_n(k)$. The same reasoning also implies that $[\mathfrak{n}_n(k), \mathfrak{b}_n(k)] = \mathfrak{n}_n(k)$, so the lower central series of $\mathfrak{b}_n(k)$ is given by $(\mathfrak{b}_n(k))_0 = \mathfrak{b}_n(k)$ and $(\mathfrak{b}_n(k))_i = \mathfrak{n}_n(k)$ for all $i > 0$.

For an upper-triangular matrix M with row- i , column- j entry $X_{i,j}$, let the m^{th} diagonal of X be the list of entries $(X_{m,i} : i = 1, 2, \dots, n - m + 1)$ for each $m \in \{1, 2, \dots, n\}$, and let V_m denote the space of upper-triangular matrices with i^{th} diagonal equal to 0 for every $i \in \{1, 2, \dots, m\}$; for convenience, we take $V_m = 0$ for all $m \geq n$. We have that $\mathfrak{n}_n(k) = V_1$, and one readily checks that $[V_i, V_j] \subset V_{i+j}$ for each $i, j \in \{1, 2, \dots, n\}$. It follows that the derived series of $\mathfrak{b}_n(k)$ satisfies $(\mathfrak{b}_n(k))^0 = \mathfrak{b}_n(k)$ and $(\mathfrak{b}_n(k))^i \subset V_{2^i}$ for each $i \in \{1, 2, \dots, n\}$, so $(\mathfrak{b}_n(k))^i = 0$ for each $i > \log_2 n$. It also follows that the lower central series of $\mathfrak{n}_n(k)$ satisfies $(\mathfrak{n}_n(k))_0 = \mathfrak{n}_n(k)$, $(\mathfrak{n}_n(k))_i \subset V_{i+1}$ for each $i \in \{1, \dots, n - 1\}$, so $(\mathfrak{n}_n(k))_i = 0$ for each $i > n - 1$. ♣

The next lemma establishes a few nice properties of solvable Lie algebras:

Lemma 15. *Let \mathfrak{g} be an abstract Lie algebra, and let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ be ideals. Then \mathfrak{g} is solvable if and only if \mathfrak{a} , viewed as a Lie algebra, is solvable and $\mathfrak{g}/\mathfrak{a}$ is solvable. If $\mathfrak{a}, \mathfrak{b}$ are solvable, then $\mathfrak{a} + \mathfrak{b}$ is solvable.*

Proof. For the first statement, one checks by induction that $\mathfrak{a}^n \subset \mathfrak{g}^n$, so \mathfrak{a} is solvable if \mathfrak{g} is. The way one computes a Lie bracket of elements in $(\mathfrak{g}/\mathfrak{a})^n$ is by taking representatives in \mathfrak{g} , computing the Lie bracket in \mathfrak{g} , and then passing to the quotient $\mathfrak{g}/\mathfrak{a}$. Thus, if $\mathfrak{g}^n = 0$ then $(\mathfrak{g}/\mathfrak{a})^n = 0$. If $\mathfrak{a}^{n_1} = 0$ and $(\mathfrak{g}/\mathfrak{a})^{n_2} = 0$, then $\mathfrak{g}^{n_2} \subset \mathfrak{a}$, so $(\mathfrak{g}^{n_2})^{n_1} = 0$. For the second statement, it suffices by Lemma 12 to show that $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ is solvable, but this follows from the first statement. ♠

Thus far we have discussed solvable Lie algebras, which are in some sense close to being abelian. We now introduce two new types of Lie algebras that are essentially the opposite in that they are very far from being abelian:

Definition 16. A Lie algebra is said to be *semisimple* if it has no nonzero solvable ideals and is said to be *simple* if it is nonabelian and has no nonzero proper ideals.

If \mathfrak{g} is a simple Lie algebra with a nonzero solvable ideal, then that ideal must be \mathfrak{g} , but then $0 \subsetneq [\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$, a contradiction implying that \mathfrak{g} is necessarily semisimple (so the terminology makes sense). Note that if \mathfrak{g} is a semisimple Lie algebra, then the center $Z_{\mathfrak{g}}$ (see part (b) of Example 5) is trivial, because it is necessarily a nilpotent ideal of \mathfrak{g} . But since $Z_{\mathfrak{g}} = \ker \text{ad}(\mathfrak{g})$ (see part (b) of Example 11), we deduce that the adjoint representation is faithful for semisimple Lie algebras. This proves Ado's Theorem (see Section 3) for semisimple Lie algebras.

It is natural to wonder whether there is some link between simplicity of Lie algebras and simplicity of representations of Lie algebras (given the similarity in terminology), and indeed there is. Although we omit the proof for the sake of brevity, a finite-dimensional Lie algebra is semisimple if and only if its adjoint representation is semisimple. In particular, a Lie algebra is simple if and only if it is nonabelian and its adjoint representation is irreducible (this is rather easy to prove: invariant subspaces of the adjoint representation are none other than ideals).

It follows from Lemma 15 that every finite-dimensional Lie algebra \mathfrak{g} has a unique largest solvable ideal; this ideal is called the *radical* and is denoted by $\text{rad}(\mathfrak{g})$. In this setting, note that a Lie algebra \mathfrak{g} is semisimple if and only if $\text{rad}(\mathfrak{g}) = 0$. Consequently, one might expect that quotienting out by the radical will turn a Lie algebra into a semisimple one, and this is indeed true in the finite-dimensional setting:

Lemma 17. *Let \mathfrak{g} be a finite-dimensional Lie algebra, and let $\mathfrak{a} \subset \mathfrak{g}$ be a solvable ideal. Then $\mathfrak{g}/\mathfrak{a}$ is semisimple if and only if $\mathfrak{a} = \text{rad}(\mathfrak{g})$.*

Proof. For the forward direction, if $\mathfrak{a} \neq \text{rad}(\mathfrak{g})$, then consider the nonzero ideal $\text{rad}(\mathfrak{g})/\mathfrak{a} \subset \mathfrak{g}/\mathfrak{a}$. Because $\text{rad}(\mathfrak{g})$ is solvable, we have by Lemma 15 that $\text{rad}(\mathfrak{g})/\mathfrak{a}$ is solvable, a contradiction implying that $\mathfrak{a} = \text{rad}(\mathfrak{g})$.

For the reverse direction, note that the preimage of an ideal under a Lie algebra homomorphism is also an ideal. Any nonzero solvable ideal of $\text{rad}(\mathfrak{g})$ has as its preimage under the projection map $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\text{rad}(\mathfrak{g})$ a solvable ideal of \mathfrak{g} containing $\text{rad}(\mathfrak{g})$. By the definition of $\text{rad}(\mathfrak{g})$, any such ideal must be equal to $\text{rad}(\mathfrak{g})$. It follows that $\mathfrak{g}/\text{rad}(\mathfrak{g})$ has no nonzero solvable ideals and is therefore semisimple. ♠

To conclude this section, note that by analogy with the radical $\text{rad}(\mathfrak{g})$ of a finite-dimensional Lie algebra \mathfrak{g} we also have the *nilradical* $\text{nil}(\mathfrak{g})$, which is defined to be the largest nilpotent ideal of \mathfrak{g} . The nilradical is well-defined because, just as we had with solvable ideals, the sum of any two nilpotent ideals is also nilpotent. Note that $\text{nil}(\mathfrak{g}) \subset \text{rad}(\mathfrak{g})$.

2.2. Properties of Lie Algebra Representations. An extremely useful lemma regarding representations in general (not just of Lie algebras) is Schur's Lemma:

Lemma 18 (Schur's Lemma). *Let $\mu_1 : \mathfrak{g} \rightarrow \mathfrak{gl}(V_1)$ and $\mu_2 : \mathfrak{g} \rightarrow \mathfrak{gl}(V_2)$ be irreducible Lie algebra representations, and let $\phi : V_1 \rightarrow V_2$ be a homomorphism of \mathfrak{g} -representations. Then either $\phi = 0$ or ϕ is an isomorphism. If μ_1, μ_2 are complex, then there exists a scalar $\lambda \in \mathbb{C}$ with $\phi(v) = \lambda v$ for all $v \in V_1$.*

Proof. The first statement is obvious upon observing that $\ker \phi$ and $\text{im } \phi$ are \mathfrak{g} -invariant subspaces of V_1 and V_2 , respectively. For the second statement, the fact that we are dealing with complex representations means that ϕ has an eigenvalue λ , so $\ker(\phi - \lambda \cdot \text{id}) \subset V_1$ is a nonzero \mathfrak{g} -invariant subspace of V_1 and must therefore be all of V_1 . ♠

In the case of finite-dimensional representations, we can use Schur's Lemma to prove that the condition of semisimplicity is in fact equivalent to the condition of complete reducibility:

Proposition 19. *Let $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a finite-dimensional representation. Then μ is semisimple if and only if (A) every subrepresentation of V is also semisimple if and only if (B) V is completely reducible, in the sense that for some n there exist irreducible subrepresentations $\mu_i : \mathfrak{g} \rightarrow \mathfrak{gl}(V_i)$ for $i \in \{1, 2, \dots, n\}$ satisfying $\mu = \bigoplus_{i=1}^n \mu_i$.*

Proof. For equivalence (A), suppose μ is semisimple, and let W be a subrepresentation of V . If $U \subset W$ is a \mathfrak{g} -invariant subspace, then there exists a \mathfrak{g} -invariant subspace $U'' \subset V$ with $V = U \oplus U''$, and so $W = U \oplus U'$ where $U' = U'' \cap W$. It follows that W gives rise to a semisimple representation of \mathfrak{g} . The other direction is obvious.

For equivalence (B), the forward direction follows immediately from inducting on the dimension of V : if V is not irreducible, then split V as $V = W \oplus W'$ for \mathfrak{g} -invariant subspaces $W, W' \subset V$ and then repeat. As for the reverse direction, take $W \subset V$ a \mathfrak{g} -invariant subspace. Applying Schur's Lemma to the composite map $W \hookrightarrow V \twoheadrightarrow V_i$ for each i yields that $W = \bigoplus_{j=1}^k V_{i_j}$ for some subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. Then $V = W \oplus W'$, where W' is the direct sum of V_i over all $i \notin \{i_1, \dots, i_k\}$. ♠

We conclude this section with a nice result that links representations of a real Lie algebra and its complexification (see part (g) of Example 5):

Proposition 20. *Every complex representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ extends uniquely to a complex representation $\mu_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{gl}(V)$, and the map $\mu \mapsto \mu_{\mathbb{C}}$ gives a bijection between complex representations of \mathfrak{g} and complex representations of $\mathfrak{g}_{\mathbb{C}}$. This bijection preserves irreducibility.*

Proof. Given a complex representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, we would like to define a complex representation $\mu_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{gl}(V)$ that extends μ . Since $\mu_{\mathbb{C}}$ must be a representation of Lie algebras, we require that $\mu_{\mathbb{C}}$ be a homomorphism of

Lie algebras. Thus for $X, Y \in \mathfrak{g}$, we must have that $\mu_{\mathbb{C}}(X + iY) = \mu_{\mathbb{C}}(X) + i\mu_{\mathbb{C}}(Y) = \mu(X) + i\mu(Y)$, implying that $\mu_{\mathbb{C}}$ is unique, if it exists. To check that $\mu_{\mathbb{C}}$ exists, we need to verify that the map $\mu_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{gl}(V)$ defined by sending $X + iY \mapsto \mu(X) + i\mu(Y)$ commutes with the Lie bracket. Notice that

$$\begin{aligned} \mu_{\mathbb{C}}([X + iY, X' + iY']) &= \mu_{\mathbb{C}}([X, X'] - [Y, Y'] + i([X, Y'] + [Y, X'])) = \\ &= \mu([X, X']) - \mu([Y, Y']) + i(\mu([X, Y']) + \mu([Y, X'])) = \\ &= [\mu(X), \mu(X')] - [\mu(Y), \mu(Y')] + i([\mu(X), \mu(Y')] + [\mu(Y), \mu(X')]), \end{aligned}$$

and that we also have

$$\begin{aligned} [\mu_{\mathbb{C}}(X + iY), \mu_{\mathbb{C}}(X' + iY')] &= [\mu(X) + i\mu(Y), \mu(X') + i\mu(Y')] = \\ &= [\mu(X), \mu(X')] - [\mu(Y), \mu(Y')] + i([\mu(X), \mu(Y')] + [\mu(Y), \mu(X')]), \end{aligned}$$

which establishes the desired compatibility, so $\mu_{\mathbb{C}}$ exists and is uniquely defined.

Consider the map Φ from complex representations of \mathfrak{g} to complex representations of $\mathfrak{g}_{\mathbb{C}}$ defined by $\mu \mapsto \mu_{\mathbb{C}}$, and consider the map Ψ in the reverse direction defined by restriction. We shall prove that these maps are mutually inverse. Indeed, notice that the restriction of $\mu_{\mathbb{C}}$ to \mathfrak{g} is by its very definition equal to μ , which proves that the map $\Psi \circ \Phi = \text{id}$. Now, given a complex representation $\tilde{\mu} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{gl}(V)$, we observe that for $X, Y \in \mathfrak{g}$ we have $\tilde{\mu}(X + iY) = \Psi(\tilde{\mu})(X) + i\Psi(\tilde{\mu})(Y) = \Phi(\Psi(\tilde{\mu}))(X + iY)$, implying that $\Phi \circ \Psi = \text{id}$. We have thus proven the desired bijection.

Finally, if $W \subset V$ is an invariant subspace for the representation $\mu_{\mathbb{C}}$, then $XW \subset W$ for all $X \in \mathfrak{g}$, implying that W is an invariant subspace for the representation μ . Thus, if μ is irreducible, so that its only invariant subspaces are 0 and V , then $\mu_{\mathbb{C}}$ must also be irreducible. Now suppose $W \subset V$ is an invariant subspace for the representation μ , which implies that $XW \subset W$ for all $X \in \mathfrak{g}$. Then for any $X, Y \in \mathfrak{g}$ we have that $(X + iY)W \subset XW + iYW = XW + YW \subset W + W = W$. Thus, W is an invariant subspace for the representation $\mu_{\mathbb{C}}$, so if $\mu_{\mathbb{C}}$ is irreducible, meaning that its only invariant subspaces are 0 and V , then μ must also be irreducible. ♠

2.3. Three Key Implements. In this section, we present three key implements that we will make extensive use of in the proof of Ado's Theorem.

2.3.1. The Universal Enveloping Algebra. Observe that we do not *a priori* have a notion of multiplication of elements in a Lie algebra \mathfrak{g} . However, if we work with the image of \mathfrak{g} under a representation, we do have a notion of multiplication (given by composition of endomorphisms), and in particular, we do have a notion of multiplication for linear Lie algebras (subalgebras of $\mathfrak{gl}(V)$). Because this notion of multiplication depends on the choice of representation ρ , it would be nice if we could construct a "universal" object that somehow simultaneously

captures all of these notions of multiplication. It is for this reason that we introduce the universal enveloping algebra:

Definition 21. Let \mathfrak{g} be a Lie algebra over a field k (here $k = \mathbb{R}$ or $k = \mathbb{C}$). The *universal enveloping algebra* $U(\mathfrak{g})$ of \mathfrak{g} is the unital associative algebra over k , generated by the symbols $\iota(X)$ for $X \in \mathfrak{g}$ subject to the relations $\iota([X, Y]) = \iota(X)\iota(Y) - \iota(Y)\iota(X)$ for all $X, Y \in \mathfrak{g}$.

Note that for a nonzero Lie algebra \mathfrak{g} , the universal enveloping algebra is always infinite-dimensional. The Poincaré-Birkhoff-Witt Theorem (which we will not prove for the sake of brevity) gives us a nice basis for the universal enveloping algebra:

Theorem 22 (Poincaré-Birkhoff-Witt). *Let the list $(X_1, X_2, \dots, X_n) \subset \mathfrak{g}$ be a basis for \mathfrak{g} . Then the monomials of the form $X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}$ for nonnegative integers i_1, i_2, \dots, i_n form a basis of $U(\mathfrak{g})$.*

It follows immediately from the Poincaré-Birkhoff-Witt Theorem that the Lie algebra homomorphism $\iota : \mathfrak{g} \rightarrow U(\mathfrak{g})$ defined by sending X to the symbol $\iota(X)$ is injective. Therefore, it makes sense to drop the ι 's when referring to elements of $U(\mathfrak{g})$, and throughout the rest of this article, we shall simply write X for $\iota(X)$. The next theorem demonstrates that the universal enveloping algebra lives up to its name:

Theorem 23. *Let A be a unital associative algebra over k (where $k = \mathbb{R}$ or $k = \mathbb{C}$), and let $\mu : \mathfrak{g} \rightarrow A$ be a map of k -vector spaces satisfying $\mu([X, Y]) = \mu(X)\mu(Y) - \mu(Y)\mu(X)$ for all $X, Y \in \mathfrak{g}$. Then there exists a unique extension of μ to a map $\tilde{\mu} : U(\mathfrak{g}) \rightarrow A$ of associative algebras such that $\mu = \tilde{\mu} \circ \iota$.*

Proof. We simply define $\tilde{\mu}(X) = \mu(X)$ for all $X \in \mathfrak{g}$ and extend multiplicatively to all of $U(\mathfrak{g})$. Then $\tilde{\mu}$ is well-defined because it vanishes on the defining relations of $U(\mathfrak{g})$: indeed, $\tilde{\mu}([X, Y]) = \mu([X, Y]) = \mu(X)\mu(Y) - \mu(Y)\mu(X) = \tilde{\mu}(X)\tilde{\mu}(Y) - \tilde{\mu}(Y)\tilde{\mu}(X) = \tilde{\mu}(XY - YX)$. That $\tilde{\mu}$ is a map of associative algebras and that $\mu = \tilde{\mu} \circ \iota$ are evident from the construction. \spadesuit

The universal enveloping algebra is one of the key tools that we will employ in the proof of Ado's Theorem (see Section 3).

2.3.2. Engel's Theorem and Corollaries. The second key implement that we will be using in the proof of Ado's Theorem is Engel's Theorem, which tells us when a finite-dimensional representation, by nilpotent operators, of a Lie algebra \mathfrak{g} has the property that every element of \mathfrak{g} acts by a strictly upper-triangular matrix. We now give a proof of this theorem:

Theorem 24 (Engel's Theorem). *Let V be a finite-dimensional vector space, and let \mathfrak{g} be a (necessarily finite-dimensional) Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ with the*

property that X acts nilpotently on V for every $X \in \mathfrak{g}$. Then there exists a basis of V with respect to which the matrix of every $X \in \mathfrak{g}$ is strictly upper-triangular.

Proof. By a standard linear algebra argument, it suffices to show that there exists a nonzero vector $v \in V$ with $X \cdot v = 0$ for all $X \in \mathfrak{g}$. We proceed by induction on $\dim \mathfrak{g}$; noting that the theorem holds trivially in the base case where $\dim \mathfrak{g} = 0$. Choose a codimension-1 ideal $\mathfrak{a} \subset \mathfrak{g}$; one way of doing this is by taking \mathfrak{a} to be any maximal element among proper subalgebras of \mathfrak{g} . Indeed, let \mathfrak{a} be such a subalgebra. Then since \mathfrak{a} acts nilpotently on V , we have that $\text{ad}(\mathfrak{a})$ acts nilpotently on $\mathfrak{gl}(V)$ and hence on $\mathfrak{g}/\mathfrak{a}$. By the induction hypothesis (for the proof of Engel's Theorem), there exists $v' \in \mathfrak{g} \setminus \mathfrak{a}$ such that $X \cdot v' \in \mathfrak{a}$ for all $X \in \mathfrak{a}$. But then $\text{span}(v', \mathfrak{a})$ is a Lie subalgebra of \mathfrak{g} properly containing \mathfrak{a} as an ideal, so $\text{span}(v', \mathfrak{a}) = \mathfrak{g}$ and the ideal \mathfrak{a} has codimension 1 in \mathfrak{g} .

Now, by the induction hypothesis, the subspace $W \subset V$ of vectors $w \in V$ with $X \cdot w = 0$ for all $X \in \mathfrak{a}$ is nonzero. If $Y \in \mathfrak{g} \setminus \mathfrak{a}$, then $\mathfrak{g} = \text{span}(\mathfrak{a}, Y)$, so to prove the theorem it suffices to show that there exists $w \in W$ with $Y \cdot w = 0$. Notice that for any $X \in \mathfrak{a}$ and $w' \in W$ we have $X \cdot (Y \cdot w') = Y \cdot (X \cdot w') + [X, Y] \cdot w' = 0 + 0 = 0$, so $Y \cdot w' \in W$, implying that $Y \cdot W \subset W$. But since the action of Y on V is nilpotent, the action of Y on W is also nilpotent, implying that there exists $w \in W$ with $Y \cdot w = 0$, and this is the desired result. ♠

We will also require the following corollaries of Engel's Theorem in the proof of Ado's Theorem:

Corollary 25. *Let V be a finite-dimensional vector space, let \mathfrak{g} be a (necessarily finite-dimensional) Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, and let $\mathfrak{a}, \mathfrak{b}$ be ideals of \mathfrak{g} such that $\mathfrak{b} \subset [\mathfrak{a}, \mathfrak{g}]$. If $[\mathfrak{a}, \mathfrak{b}]$ is nilpotent, so is \mathfrak{b} .*

Proof. Take $X \in \mathfrak{b}$. To show that X is nilpotent, it suffices to show that $\text{Tr}(X^m) = 0$ for each positive integer m . Indeed, notice that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of X with repetition (say we are working over \mathbb{C}), then $\text{Tr}(X^m) = \lambda_1^m + \dots + \lambda_n^m$, so by Newton's identities, the characteristic polynomial of X is x^n , implying that X is nilpotent by the Cayley-Hamilton Theorem.

It further suffices to show that $\text{Tr}([Y, Z]X^{m-1}) = 0$ for $Y \in \mathfrak{a}$ and $Z \in \mathfrak{g}$, because $\mathfrak{b} \subset [\mathfrak{a}, \mathfrak{g}]$. But then we have that

$$\text{Tr}([Y, Z]X^{m-1}) = -\text{Tr}(Z[Y, X^{m-1}]) = -\sum_{i=1}^{m-1} \text{Tr}(ZX^{i-1}[Y, X]X^{m-i-1}),$$

and each trace in the above sum is 0. Indeed, suppose $X_1, X_2, \dots, X_\ell \in \mathfrak{g}$ with at least one X_i being an element of the nilpotent ideal $[\mathfrak{a}, \mathfrak{b}]$. Then consider the chain of subspaces $V_i = [\mathfrak{a}, \mathfrak{b}]^i \cdot V$ (here, the superscript i does *not* refer to the derived series but to i -fold multiplication by elements in $[\mathfrak{a}, \mathfrak{b}]$). Applying Engel's Theorem to the nilpotent ideal $[\mathfrak{a}, \mathfrak{b}]$ yields that $V_j = 0$ for some j . Each V_i is \mathfrak{g} -invariant because $[\mathfrak{a}, \mathfrak{b}]$ is an ideal, so $X_1 X_2 \dots X_\ell$ sends V_i to V_{i+1} for

each i . Thus, $X_1 X_2 \cdots X_\ell$ is nilpotent and has 0 trace. It follows that \mathfrak{b} is nilpotent, as desired. ♠

Corollary 26. *Let \mathfrak{g} be a finite-dimensional Lie algebra, and let $\mathfrak{a} \subset \mathfrak{g}$ be a solvable ideal. Then $\text{ad}(X)(\mathfrak{a}) \subset \text{nil}(\mathfrak{g})$ for every $X \in \mathfrak{g}$.*

Proof. It suffices to show that the ideal $\text{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. We first deal with the linear case: suppose $\mathfrak{g} \subset \mathfrak{gl}(V)$ for some finite-dimensional vector space V . Suppose for some i we have $\text{rad}(\mathfrak{g})^{i+1} \cap [\mathfrak{g}, \mathfrak{g}]$ is nilpotent (here the superscript $i+1$ does refer to the derived series); notice that such an i exists because $\text{rad}(\mathfrak{g})^n = 0$ for n sufficiently large. Then the smaller ideal $[\text{rad}(\mathfrak{g})^i, [\text{rad}(\mathfrak{g})^i, \mathfrak{g}]]$ is nilpotent, so by Corollary 25, $[\text{rad}(\mathfrak{g})^i, \mathfrak{g}]$ is nilpotent, and applying Corollary 25 to the smaller ideal $[\mathfrak{g}, \text{rad}(\mathfrak{g})^i \cap [\mathfrak{g}, \mathfrak{g}]]$ yields that $\text{rad}(\mathfrak{g})^i \cap [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. It follows by induction that $\text{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Otherwise, if \mathfrak{g} is an abstract Lie algebra, then pass to the image $\text{ad}(\mathfrak{g})$ of \mathfrak{g} under the adjoint representation, which is solvable by Lemma 15. The previous paragraph shows that the image of $\text{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ is nilpotent, so since the kernel of the adjoint representation is the manifestly nilpotent ideal $Z_{\mathfrak{g}}$, we have that $\text{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ must itself be nilpotent. ♠

2.3.3. *Levi's Theorem.* It follows from Lemma 17 that we have a short exact sequence of Lie algebras

$$0 \longrightarrow \text{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\text{rad}(\mathfrak{g}) \longrightarrow 0.$$

It is natural to ask when the above short exact sequence splits. The answer to this question is provided by Levi's Theorem, which is the third and final implement that we will require for the proof of Ado's Theorem:

Theorem 27 (Levi's Theorem). *Let \mathfrak{g} be a (real or complex) Lie algebra. Then there is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, called a Levi subalgebra, giving a vector space decomposition $\mathfrak{g} = \text{rad}(\mathfrak{g}) \oplus \mathfrak{h}$.³*

Proof. We proceed by induction on $\dim \mathfrak{g}$. When $\dim \mathfrak{g} = 0$ there is nothing to prove, so suppose $\dim \mathfrak{g} > 0$. If $\text{rad}(\mathfrak{g})$ contains a nonzero proper ideal \mathfrak{a} , then by the induction hypothesis there exists a Levi subalgebra $\mathfrak{h}' \subset \mathfrak{g}/\mathfrak{a}$ for $\text{rad}(\mathfrak{g})/\mathfrak{a}$. Since \mathfrak{h}' is semisimple, we have that $\text{rad}(\mathfrak{h}') = 0$. Let \mathfrak{h}'' denote the preimage of \mathfrak{h}' under the projection map $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{a}$. Then since $\mathfrak{h}' = \mathfrak{h}''/\mathfrak{a}$, we have that $\text{rad}(\mathfrak{h}'') = \mathfrak{a}$. Noting that $\dim \mathfrak{h}'' < \dim \mathfrak{g}$, because $\dim \text{rad}(\mathfrak{g})/\mathfrak{a} > 0$, we have by the induction hypothesis that there exists a Levi subalgebra $\mathfrak{h} \subset \mathfrak{h}''$ for \mathfrak{a} . Then $\mathfrak{h} \subset \mathfrak{g}$ is also a Levi subalgebra for $\text{rad}(\mathfrak{g})$.

We may therefore assume that $\text{rad}(\mathfrak{g})$ contains no nonzero proper ideals. If $\text{rad}(\mathfrak{g})$ is nonabelian, then $[\text{rad}(\mathfrak{g}), \text{rad}(\mathfrak{g})] \subset \text{rad}(\mathfrak{g})$ is a nonzero ideal, and it is proper because $\text{rad}(\mathfrak{g})$ is solvable. We thus have a contradiction implying

³Warning: this decomposition is not compatible with the Lie bracket operation.

that $\text{rad}(\mathfrak{g})$ is abelian. Since $[\text{rad}(\mathfrak{g}), \mathfrak{g}] \subset \text{rad}(\mathfrak{g})$, either $[\text{rad}(\mathfrak{g}), \mathfrak{g}] = 0$ or $[\text{rad}(\mathfrak{g}), \mathfrak{g}] = \mathfrak{g}$. If the former is true, then $\text{rad}(\mathfrak{g}) = Z_{\mathfrak{g}}$ because $Z_{\mathfrak{g}}$ is nilpotent and thus solvable. Since $Z_{\mathfrak{g}} = \ker \text{ad}$, the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ descends to a representation $\mu : \mathfrak{g}/\text{rad}(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{g})$. We now invoke a fact that we will not prove because its proof requires much structure theory that is not otherwise relevant to the present exposition:

Theorem 28. *Let $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra \mathfrak{g} . Then μ is semisimple.*

By Theorem 28 and because $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is semisimple, the \mathfrak{g} -invariant subspace $\text{rad}(\mathfrak{g}) \subset \mathfrak{g}$ has a \mathfrak{g} -invariant complement \mathfrak{h} , so we obtain a decomposition of Lie algebras $\mathfrak{g} = \text{rad}(\mathfrak{g}) \oplus \mathfrak{h}$, with \mathfrak{h} being the desired Levi subalgebra. We may therefore assume that $[\text{rad}(\mathfrak{g}), \mathfrak{g}] = \mathfrak{g}$.

To finish the proof, consider the representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{gl}(\mathfrak{g}))$ defined by $X \cdot \xi = \text{ad}(X) \circ \xi - \xi \circ \text{ad}(X)$ for all $X \in \mathfrak{g}$ and $\xi \in \mathfrak{gl}(\mathfrak{g})$, and consider the vector subspaces $A, B, C \subset \mathfrak{gl}(\mathfrak{g})$ defined by

$$\begin{aligned} A &= \{\text{ad}(X) : X \in \text{rad}(\mathfrak{g})\}, \\ B &= \{\xi \in \mathfrak{gl}(\mathfrak{g}) : \xi(\mathfrak{g}) \subset \text{rad}(\mathfrak{g}) \text{ and } \xi(\text{rad}(\mathfrak{g})) = 0\}, \\ C &= \{\xi \in \mathfrak{gl}(\mathfrak{g}) : \xi(\mathfrak{g}) \subset \text{rad}(\mathfrak{g}) \text{ and } \xi|_{\text{rad}(\mathfrak{g})} \text{ is multiplication by a scalar}\}, \end{aligned}$$

Notice that $A \subset B$ because $\text{rad}(\mathfrak{g})$ is abelian and that $B \subset C$ by construction. An application of the Jacobi identity tells us that A is \mathfrak{g} -invariant (with respect to the representation μ), and B, C are evidently \mathfrak{g} -invariant. Observe that $C/B = k$ as vector spaces (where $k = \mathbb{R}$ or $k = \mathbb{C}$ depending on whether \mathfrak{g} is real or complex) via the map sending $\xi \in C$ to the scalar λ by which ξ acts on $\text{rad}(\mathfrak{g})$. But because the Lie bracket on C/B is trivial (since multiplication by scalars is commutative), we have that $C/B = k$ as Lie algebras. One can likewise check that $\mathfrak{g} \cdot C \subset B$ and that $\text{rad}(\mathfrak{g}) \cdot C \subset A$, from which we deduce that C/A and C/B can be given the structure of $\mathfrak{g}/\text{rad}(\mathfrak{g})$ -representation.

Consider the surjective map of vector spaces $\phi : C/A \rightarrow C/B = k$ induced by the identity map on C . Since for any $X + \text{rad}(\mathfrak{g}) \in \mathfrak{g}/\text{rad}(\mathfrak{g})$ and $\xi + A \in C/A$ we have $\phi((X + \text{rad}(\mathfrak{g})) \cdot (\xi + A)) = X \cdot \xi + B = (X + \text{rad}(\mathfrak{g})) \cdot (\xi + B)$, the map ϕ is compatible with the action of $\mathfrak{g}/\text{rad}(\mathfrak{g})$. Because $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is semisimple, Theorem 28 tells us that ϕ splits, i.e. it has a right inverse $\psi : C/B \rightarrow C/A$. Let $\xi \in C$ be a preimage of $\psi(1) \in C/A$ so that $\mathfrak{g} \cdot \xi \subset A$, and observe that $\xi|_{\text{rad}(\mathfrak{g})} = \text{id}$.

Consider the subspace $\mathfrak{h} \subset \mathfrak{g}$ defined by $\mathfrak{h} = \{X \in \mathfrak{g} : X \cdot \xi = 0\}$. Clearly, \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . If $X \in \text{rad}(\mathfrak{g}) \cap \mathfrak{h}$, then $0 = X \cdot \xi = -\text{ad}(X)$, implying that $\text{ad}(X) = 0$ and that the subspace $\{Y : Y = cX \text{ for some } c \in k\}$ is a nonzero ideal of \mathfrak{g} contained in $\text{rad}(\mathfrak{g})$, contradicting our assumption that $\text{rad}(\mathfrak{g})$ contains no such ideals. Moreover, if $X \in \mathfrak{g}$, then $X \cdot \xi \in A$ so $X \cdot \xi = \text{ad}(Y)$ for some

$Y \in \text{rad}(\mathfrak{g})$. But then $(X + Y) \cdot \xi = \text{ad}(Y) - \text{ad}(Y) = 0$, so $X + Y \in \mathfrak{h}$. It follows that $\mathfrak{g} = \text{rad}(\mathfrak{g}) + \mathfrak{h}$, so in fact $\mathfrak{g} = \text{rad}(\mathfrak{g}) \oplus \mathfrak{h}$. ♠

Note that if $\mathfrak{h} \subset \mathfrak{g}$ is a Levi subalgebra for $\text{rad}(\mathfrak{g})$, then \mathfrak{h} cannot contain any nonzero solvable ideals, so \mathfrak{h} is semisimple.

3. PROVING ADO'S THEOREM

In its purest form, Ado's Theorem states that every finite-dimensional (real or complex) Lie algebra is linear, in the sense that it has a faithful finite-dimensional representation. But as with many theorems in mathematics, we will find it convenient to work toward a stronger statement. We first need to define a special kind of Lie algebra representation:

Definition 29. Let $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a Lie algebra representation. Then μ is said to be a *nilrepresentation* if $\mu(X)$ is a nilpotent element of $\mathfrak{gl}(V)$ for all $X \in \text{nil}(\mathfrak{g})$.

Note that by Engel's Theorem, a nilrepresentation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ satisfies the property that $\mu(\mathfrak{g})^m = 0$ for some positive integer m (indeed, $m = \dim V$ works). In Sections 3.1–3.3, we will prove the following Ado-type theorem about nilrepresentations of Lie algebras:

Theorem 30. *Every finite-dimensional (real or complex) Lie algebra has a faithful finite-dimensional nilrepresentation.*

We follow the strategy detailed in [4] (which in turn follows the proof presented in [1]); namely, we first tackle the nilpotent and solvable cases (in that order; see Sections 3.1 and 3.2) before dealing with the general case (see Section 3.3). Before we get on with the proof, we observe that relaxing the conditions of finite-dimensionality renders the proof of Ado's Theorem almost too easy:

Theorem 31. *Every Lie algebra has a faithful representation.*

Proof. Let \mathfrak{g} be a Lie algebra, and consider the map $\mu_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{gl}(U(\mathfrak{g}))$ defined by sending $X \in \mathfrak{g}$ to the map of left multiplication by X . Clearly $\mu_{\mathfrak{g}}$ is a map of vector spaces, and $\mu_{\mathfrak{g}}([X, Y])(Z) = [X, Y]Z = (XY - YX)Z$, so $\mu_{\mathfrak{g}}$ is a Lie algebra representation. It is faithful because $\mu_{\mathfrak{g}}(X)(1) = X \neq 0$ for all nonzero $X \in \mathfrak{g}$. ♠

However, the universal enveloping algebra is far from being finite-dimensional, and proving Theorem 30 will be considerably more involved.

Let \mathfrak{g} be a finite-dimensional Lie algebra. If $\mu' : \mathfrak{g} \rightarrow \mathfrak{gl}(V')$ is a finite-dimensional nilrepresentation whose restriction to $Z_{\mathfrak{g}}$ is faithful, then the representation $\mu : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, where $V = V' \oplus \mathfrak{g}$ defined by $\mu = \mu' \oplus \text{ad}$ is faithful and finite-dimensional. In fact, μ is also a nilrepresentation, because the adjoint

representation is a nilrepresentation — given any $X \in \text{nil}(\mathfrak{g})$, $Z \in \mathfrak{g}$, we have that $[X, Z] \in \text{nil}(\mathfrak{g})$ so applying $\text{ad}(X)$ sufficiently many times will yield 0. It therefore suffices to construct such a representation μ' .

3.1. The Nilpotent Case. Suppose \mathfrak{g} is nilpotent. If \mathfrak{g} is in fact abelian, consider the map $\mu' : \mathfrak{g} \rightarrow \mathfrak{gl}(V')$, where $V' = \mathfrak{g} \oplus \mathbb{C}$, defined by $X \cdot (Y, t) = (tX, 0)$ for all $X, Y \in \mathfrak{g}$ and $t \in \mathbb{C}$. Note that μ' is a map of vector spaces and that $\mu'(X)$ is nilpotent for all $X \in \mathfrak{g}$. Moreover, $\mu'([X, Y])(Z, t) = (t[X, Y], 0) = (0, 0) = X \cdot (tY, 0) - Y \cdot (tX, 0)$ and $X \cdot (0, 1) = (X, 0)$, so μ' is a faithful nilrepresentation, thus proving Theorem 30 in the abelian case.

We may now suppose that \mathfrak{g} is nonabelian, so that $Z_{\mathfrak{g}}$ is an ideal of \mathfrak{g} having strictly smaller dimension (here $\dim \mathfrak{g}$ is necessarily positive). We will proceed by induction on $\dim \mathfrak{g}$: suppose that the theorem holds for all Lie algebras of smaller dimension. The strategy will be to find a proper ideal $\mathfrak{a} \subset \mathfrak{g}$ containing $Z_{\mathfrak{g}}$, apply the induction hypothesis to \mathfrak{a} , and extend the resulting nilrepresentation to all of \mathfrak{g} in such a way that it remains faithful on \mathfrak{a} .

We first claim that we can choose the desired ideal \mathfrak{a} so that it has codimension 1. Indeed, to find such an ideal is to find a codimension-1 ideal of the (nonzero) Lie algebra $\mathfrak{g}' = \mathfrak{g}/Z_{\mathfrak{g}}$, for which it suffices to find a codimension-1 ideal of the abelian Lie algebra $\mathfrak{g}'' = \mathfrak{g}'/[\mathfrak{g}', \mathfrak{g}']$. But, as stated in part (a) of Example 5, any codimension-1 subspace of \mathfrak{g}'' will do. The claim follows, and by choosing $\mathfrak{h} \subset \mathfrak{g}$ to be an arbitrary subspace complementary to \mathfrak{a} , we obtain a vector space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$. Observe that this decomposition is not compatible with the Lie bracket, but because \mathfrak{h} is 1-dimensional, it is nonetheless an abelian Lie subalgebra of \mathfrak{g} (see part (a) of Example 5).

Now consider the universal enveloping algebra $U(\mathfrak{a})$, which gives rise to a representation $\mu_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{gl}(U(\mathfrak{a}))$ (see the proof of Theorem 31). We want to extend this representation to be defined on all of \mathfrak{g} . To do so, consider the map $\mu'' : \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h} \rightarrow \mathfrak{gl}(U(\mathfrak{a}))$ defined (on the spanning set of monomials in $U(\mathfrak{a})$) by the “Leibniz rule”

$$(Y, X) \cdot (Z_1 Z_2 \cdots Z_m) = Y Z_1 Z_2 \cdots Z_m + \sum_{i=1}^m Z_1 \cdots [X, Z_i] \cdots Z_m$$

for all $X \in \mathfrak{h}$ and $Y, Z_1, \dots, Z_m \in \mathfrak{a}$. It is clear that μ'' is a map of vector spaces; further notice that

$$\begin{aligned} & (Y, X) \cdot ((Y', X') \cdot (Z_1 Z_2 \cdots Z_m)) = \\ & Y Y' Z_1 Z_2 \cdots Z_m + [X, Y'] Z_1 Z_2 \cdots Z_m + \\ & Y \sum_{i=1}^m Z_1 \cdots [X', Z_i] \cdots Z_m + Y' \sum_{i=1}^m Z_1 \cdots [X, Z_i] \cdots Z_m + \end{aligned}$$

$$\sum_{i=1}^n Z_1 \cdots [X, [X', Z_i]] \cdots Z_n + \sum_{1 \leq i \neq j \leq n} Z_1 \cdots [X', Z_i] \cdots [X, Z_j] \cdots Z_n.$$

Applying the Jacobi identity and a symmetry argument then yields that

$$(Y, X) \cdot ((Y', X') \cdot (Z_1 Z_2 \cdots Z_m)) - (Y', X') \cdot ((Y, X) \cdot (Z_1 Z_2 \cdots Z_m)) = (YY' + [X, Y'] - Y'Y - [X', Y])(Z_1 Z_2 \cdots Z_m) = [(Y, X), (Y', X')] \cdot (Z_1 Z_2 \cdots Z_m),$$

from which we conclude that μ'' is a Lie algebra representation. By considering the action of \mathfrak{a} on $1 \in U(\mathfrak{a})$ we see that μ'' is faithful on \mathfrak{a} .

If we are to get anywhere with the above construction, we must find a way to turn μ'' into a finite-dimensional representation. For this, we turn to the induction hypothesis: there exists a faithful finite-dimensional nilrepresentation $\mu_0 : \mathfrak{a} \rightarrow \mathfrak{gl}(V_0)$ and a positive integer m with $\mu_0(\mathfrak{a})^m = 0$. Consider the quotient of $U(\mathfrak{a})$ by the two-sided ideal $I \subset U(\mathfrak{a})$ generated by degree- m monomials. Note that $U(\mathfrak{a})/I$ is finite-dimensional because it is spanned by monomials of degree strictly less than m . Also observe that the map $U(\mathfrak{a}) \rightarrow \mathfrak{gl}(V_0)$ (induced by μ_0) vanishes on I by construction and hence gives rise to a map $U(\mathfrak{a})/I \rightarrow \mathfrak{gl}(V_0)$; it follows that the composite map $\mathfrak{a} \rightarrow U(\mathfrak{a}) \rightarrow U(\mathfrak{a})/I$ is injective, because postcomposing with the map $U(\mathfrak{a})/I \rightarrow \mathfrak{gl}(V_0)$ gives the injective map $\mathfrak{a} \rightarrow \mathfrak{gl}(V_0)$. Again, by considering the action of \mathfrak{a} on $1 \in U(\mathfrak{a})/I$, we see that \mathfrak{a} acts faithfully on $U(\mathfrak{a})/I$. Since the action of \mathfrak{h} on $U(\mathfrak{a})$ also descends to an action on $U(\mathfrak{a})/I$, the representation μ'' gives rise to a finite-dimensional representation $\mu' : \mathfrak{g} \rightarrow \mathfrak{gl}(U(\mathfrak{a})/I)$ that is faithful on \mathfrak{a} .

To finish checking that μ' is the desired representation, we must show that μ' is a nilrepresentation. But this is obvious: the action of \mathfrak{a} on $U(\mathfrak{a})/I$ is nilpotent because acting by an element of \mathfrak{a} sufficiently many times will cause the degrees of all terms in an element of $U(\mathfrak{a})/I$ to exceed m , and the action of \mathfrak{h} on $U(\mathfrak{a})/I$ is nilpotent because \mathfrak{g} is nilpotent.

3.2. The Solvable Case. Now suppose \mathfrak{g} is solvable; our argument for this case will proceed in similar fashion to our argument for the nilpotent case. We will proceed by induction on $\dim \mathfrak{g}/\text{nil}(\mathfrak{g})$; for the base case, when $\dim \mathfrak{g}/\text{nil}(\mathfrak{g}) = 0$, note that this implies that \mathfrak{g} is nilpotent, so Theorem 30 holds. We may therefore suppose that $\dim \mathfrak{g}/\text{nil}(\mathfrak{g}) > 0$ and that the theorem holds for all Lie algebras $\tilde{\mathfrak{g}}$ with smaller value of $\dim \tilde{\mathfrak{g}}/\text{nil}(\tilde{\mathfrak{g}})$. It suffices to construct a finite-dimensional representation of \mathfrak{g} that is both faithful and nilpotent on $\text{nil}(\mathfrak{g})$, because $Z_{\mathfrak{g}} \subset \text{nil}(\mathfrak{g})$.

Just as we did in the nilpotent case, choose a codimension-1 ideal $\mathfrak{a} \subset \mathfrak{g}$ containing $\text{nil}(\mathfrak{g})$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace complementary to \mathfrak{a} , so that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ (recall that \mathfrak{h} is an abelian Lie subalgebra of \mathfrak{g}). Observe that $\text{nil}(\mathfrak{a}) = \text{nil}(\mathfrak{g})$.

By the induction hypothesis, there exists a finite-dimensional representation $\mu_0 : \mathfrak{a} \rightarrow \mathfrak{gl}(V_0)$ that is faithful and nilpotent on $\text{nil}(\mathfrak{a}) = \text{nil}(\mathfrak{g})$, so that by Engel's Theorem there exists a positive integer m with $\mu_0(\text{nil}(\mathfrak{g}))^m = 0$. Repeating the argument used in the nilpotent case, we obtain a representation $\mu'' : \mathfrak{g} \rightarrow \mathfrak{gl}(U(\mathfrak{a}))$ that is faithful on \mathfrak{a} and hence on $\text{nil}(\mathfrak{g})$.

In order to turn μ'' into a finite-dimensional representation, consider the two-sided ideal $I \subset U(\mathfrak{a})$ that is generated by the elements of $\text{nil}(\mathfrak{g})$ together with the elements of $\ker(U(\mathfrak{a}) \mapsto \mathfrak{gl}(V_0))$. We claim that $U(\mathfrak{a})/I^m$ is finite-dimensional. To prove this claim, observe that by the Cayley-Hamilton Theorem, for every $X \in \mathfrak{a}$ there exists a polynomial $p(X) \in U(\mathfrak{a})$ (an element of the form $\sum_{i=0}^n a_i X^i$ for scalars a_0, a_1, \dots, a_n) that vanishes under μ_0 , so that $p(X) \in I$ and $p(X)^m \in I^m$. It follows that up to an element of I^m , we can replace any monomial in $U(\mathfrak{a})$ with a monomial of bounded degree, which yields the claim.

Also observe that the map $U(\mathfrak{a}) \rightarrow \mathfrak{gl}(V_0)$ (induced by μ_0) vanishes on I , and hence on I^m , by construction and therefore gives rise to a map $U(\mathfrak{a})/I^m \rightarrow \mathfrak{gl}(V_0)$; it follows that the composite map $\mathfrak{a} \rightarrow U(\mathfrak{a}) \rightarrow U(\mathfrak{a})/I^m$ is injective, because postcomposing with the map $U(\mathfrak{a})/I^m \rightarrow \mathfrak{gl}(V_0)$ gives the injective map $\mathfrak{a} \rightarrow \mathfrak{gl}(V_0)$. By considering the action of \mathfrak{a} on $1 \in U(\mathfrak{a})/I^m$, we see that \mathfrak{a} acts faithfully on $U(\mathfrak{a})/I^m$. We claim that the action of \mathfrak{h} on $U(\mathfrak{a})$ also descends to an action on $U(\mathfrak{a})/I^m$. Indeed it follows from Corollary 26 that $\mathfrak{h} \cdot I \subset I$, and then an application of the "Leibniz rule" yields that $\mathfrak{h} \cdot I^k \subset I^k$. Thus, the representation μ'' gives rise to a finite-dimensional representation $\mu' : \mathfrak{g} \rightarrow \mathfrak{gl}(U(\mathfrak{a})/I^m)$ that is faithful on \mathfrak{a} .

To finish checking that μ' is the desired representation, we must show that μ' is a nilrepresentation. But this is obvious: the action of $\text{nil}(\mathfrak{g})$ on $U(\mathfrak{a})/I^m$ is nilpotent because $\text{nil}(\mathfrak{g})^m \cdot U(\mathfrak{a}) \subset I^m$.

3.3. The General Case. Now suppose \mathfrak{g} is not necessarily nilpotent or solvable. By Levi's Theorem, there exists a vector space decomposition $\mathfrak{g} = \text{rad}(\mathfrak{g}) \oplus \mathfrak{h}$, where \mathfrak{h} is a Levi subalgebra of \mathfrak{g} . From the solvable case, we know that there exists a finite-dimensional representation $\mu_0 : \text{rad}(\mathfrak{g}) \rightarrow \mathfrak{gl}(V_0)$ that is faithful and nilpotent on $\text{nil}(\mathfrak{g})$, so that by Engel's Theorem there exists a positive integer m with $\mu_0(\text{nil}(\mathfrak{g}))^m = 0$. One may then repeat the argument from the solvable case (essentially verbatim) to construct the desired nilrepresentation μ' . This concludes the proof of Ado's Theorem.

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