Math 295X Problem Set 2

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Problem 1

Prove that the ring of integers of the field $K = \mathbb{Q}[x]/(x^3 - x^2 - 2x - 8)$ is not monogenic; i.e., prove that $\mathcal{O}_K \neq \mathbb{Z}[\alpha]$ for any $\alpha \in \mathcal{O}_K$.

Hint: Prove the stronger theorem that if the prime 2 splits completely in a field of degree at least 3, then its ring of integers is not monogenic.

Problem 2

Verify that the two constructions given in lecture, of a binary cubic form from a cubic ring and vice versa, are mutually inverse to each other, thus completing the proof of the Delone-Faddeev correspondence.

Problem 3

Prove the Iwasawa decomposition of $\mathrm{GL}_2(\mathbb{R})$, which states that every matrix in $g \in \mathrm{GL}_2(\mathbb{R})$ may be written uniquely as a product $g = ut\theta\lambda$, where u is lower-triangular unipotent, t is diagonal of determinant 1, $\theta \in \mathrm{SO}_2(\mathbb{R})/\{\pm \mathrm{id}\}$, and λ is a scalar multiple of the identity.

Hint: Use the Gram-Schmidt process for the construction of an orthonormal basis.