Math 295X Problem Set 1

Ashvin A. Swaminathan
swaminathan@math.harvard.edu

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Problem 1

Use the Minkowski bound $\sqrt{|\Delta_{K/\mathbb{Q}}|} \geq (\pi/4)^{\deg K/2} \times (\deg K)^{\deg K}/(\deg K)!$, together with the Minowski Lattice Point Theorem (see below) to prove the Hermite-Minkowski Theorem, that there are finitely many number fields of bounded discriminant.

Theorem 1. *Let Γ be a complete lattice in a vector space V over $\mathbb{R}$, and let X be a centrally symmetric convex subset of V such that $\mathrm{Vol}(X) > 2^{\dim V} \times \mathrm{Vol}(\Gamma)$, where $\mathrm{Vol}(\Gamma)$ denotes the volume of a fundamental domain for Γ. Then X contains at least one nonzero lattice point of Γ.*

Hint: Use the Minkowski theory to show that, for number fields of fixed degree and discriminant, there are only finitely many possibilities for the minimal polynomial of a primitive element.

Problem 2

Let p be a prime. By analogy with the parametrization of quadratic rings over $\mathbb{Z}$, formulate and prove an orbit parametrization for quadratic rings over $\mathbb{Z}_p$.

Problem 3

An integer is said to be k-free if it is not divisible by any nontrivial k^{th} power. Prove that the probability that an integer is k-free is $1/\zeta(k)$.